§. **Proof of** $\boxed{\|f\|_p \lesssim \|f^\#\|_p\,,\ 1<p<\infty}$, **where** $f^\#(x) := \sup_{I \ni x} \frac{1}{|I|}\int_I |f - \langle f\rangle_I|$

→ Only two things needed:

→ **Relative Distributional Inequality**:

- F, G = non-negative functions on $\mathbb{R}$; $0<p<\infty$; $\|F\|_p<\infty$;
- $|\{x : F(x)>\lambda,\ G(x)\leq c\lambda\}| \leq a|\{x : F(x)>b\lambda\}|,\ \forall \lambda>0$

$$\Rightarrow \quad \int_{\mathbb{R}} F^p dx \leq A\int G^p dx;\quad A = \frac{1}{c^p(1-ab^{-p})}\ ;\ a<b^p$$

→ **Dyadic Maximal Function**:

$$M_{\mathcal{D}} f(x) := \sup_{I\in\mathcal{D}} \langle |f|\rangle_I \mathbb{1}_I(x)$$

- Strong p,p: $\|M_{\mathcal{D}} f\|_p \lesssim \|f\|_p$, $1<p<\infty$
- Weak $1,1$: $|\{x : M_{\mathcal{D}} f(x)>\lambda\}| \leq \frac{1}{\lambda}\|f\|_1$.

→ The goal is really to prove the following:

$$\boxed{|\{x : M_{\mathcal{D}} f(x) > 2\lambda;\ f^\#(x) < \gamma\lambda\}| \leq 2\gamma\,|\{x : M_{\mathcal{D}} f(x)>\lambda\}|\ \ \forall \lambda,\gamma>0} \quad (*)$$

$\Rightarrow |\{x : M_{\mathcal{D}} f(x) > \lambda;\ f^\#(x) < \gamma/2\,\lambda\}| \leq 2\gamma\,|\{x : M_{\mathcal{D}} f(x) > \lambda/2\}|$

$\Rightarrow$ RDI with $F = M_{\mathcal{D}} f$, $G = f^\#$; $a = 2\gamma$, $b = 1/2$, $c = \gamma/2$: $\left.\begin{array}{r}\|M_{\mathcal{D}} f\|_p \lesssim \|f^\#\|_p \\ \|f\|_p \lesssim \|M_{\mathcal{D}} f\|_p\end{array}\right\} \Rightarrow$

$$\Rightarrow \|f\|_p \lesssim \|f^\#\|_p.$$

as long as $a<b^p$, i.e. $2\gamma < \frac{1}{2^p} \Rightarrow \gamma < \frac{1}{2^{p-1}}$ But in $(*)$ we may choose γ as small as we want!

Remark: The exact same proof goes through to show that $\boxed{\|f\|_p \lesssim \|f^\#_{\mathcal{D}}\|_p}$ (just replace $f^\#$ with $f^\#_{\mathcal{D}}$ everywhere in the proof).

Proof of: $\boxed{|\{x : M_{\mathcal{D}}f(x) > 2\lambda,\ f^{\#}(x) < \gamma\lambda\}| \leq 2\gamma\, |\{x : M_{\mathcal{D}}f(x) > \lambda\}|}$ $\forall \lambda, \gamma > 0$

→ Let $E_\lambda := \{x : M_{\mathcal{D}}f(x) > \lambda\}$. We may assume $|E_\lambda| < \infty$ – otherwise, the inequality is trivial.

→ $x \in E_\lambda \Rightarrow M_{\mathcal{D}}f(x) > \lambda \Rightarrow \exists$ interval $I_x \in \mathcal{D}$ s.t. $x \in I_x$ and $\langle |f| \rangle_{I_x} > \lambda$

But: **there is a maximal such interval**: if there isn't, $\langle |f| \rangle_J > \lambda,\ \forall J \supset I_x$

If $x \geq 0$, this means: $\forall y \geq 0, \exists J \supset I_x$ s.t. $y \in J$ and $\langle |f| \rangle_J > \lambda \Rightarrow M_{\mathcal{D}}f(y) \geq \lambda, \forall y \geq 0$

(Similarly if $x < 0$) $\Rightarrow |E_\lambda| = \infty$ ↯

(This is assuming the standard dyadic grid "centered" at 0, but the same argument works for any dyadic grid on $\mathbb{R}$).

→ So let $\mathcal{J}_\lambda^{max} :=$ **$\{I_x$ maximal in the sense above: $x \in E_\lambda\}$** be the collection of all such maximal intervals.

$\Rightarrow$ Clearly $E_\lambda \subset \bigcup_{I \in \mathcal{J}_\lambda^{max}} I$. Conversely: if $I \in \mathcal{J}_\lambda^{max}$, $\forall x \in I : \langle |f| \rangle_I > \lambda \Rightarrow M_{\mathcal{D}}f(x) > \lambda \Rightarrow x \in E_\lambda$

$\Rightarrow \mathcal{J}_\lambda^{max}$ is a countable collection of disjoint intervals (by maximality) $\Rightarrow \bigcup_{I \in \mathcal{J}_\lambda^{max}} I \subset E_\lambda$

$\Rightarrow E_\lambda$ is the disjoint countable union of the maximal dyadic intervals $I \in \mathcal{J}_\lambda^{max}$ with $\langle |f| \rangle_I > \lambda$

$\Rightarrow$ Suffices to prove: $\boxed{|\{x \in I : M_{\mathcal{D}}f(x) > 2\lambda;\ f^{\#}(x) < \gamma\lambda\}| \leq 2\gamma |I|,\ \forall I \in \mathcal{J}_\lambda^{max}}$

(because:) $\{M_{\mathcal{D}}f(x) > 2\lambda;\ f^{\#}(x) < \gamma\lambda\} \subseteq \{M_{\mathcal{D}}f(x) > 2\lambda\} \subseteq \{M_{\mathcal{D}}f(x) > \lambda\} = \bigcup_{I \in \mathcal{J}_\lambda^{max}} I$

→ So let $I \in \mathcal{J}_\lambda^{max}$ and assume there exists $x_0 \in I$ such that $f^{\#}(x_0) < \gamma\lambda$ (otherwise, the set on the left is $\emptyset$ and the inequality is trivial).

→ **Claim:** **Let $x \in I$ with $M_{\mathcal{D}}f(x) > 2\lambda$. Then $M_{\mathcal{D}}(f\mathbb{1}_I)(x) > 2\lambda$.**

$M_{\mathcal{D}}f(x) = \sup_{\substack{J \in \mathcal{D} \\ J \ni x}} \langle |f| \rangle_J > 2\lambda$; but $I \cap J \neq \emptyset \Rightarrow$ only taking sup over $J \supsetneq I$ or $J \subset I$.

But if $J \supsetneq I$, then $\langle |f| \rangle_J < \lambda$ (by maximality of I in E_λ), so really

$$M_{\mathcal{D}}f(x) = \underbrace{\sup_{\substack{J \subset I \\ J \ni x}} \langle |f| \rangle_J}_{= \sup_{\substack{J \subset I \\ J \ni x}} \langle |f\mathbb{1}_I| \rangle_J} > 2\lambda \qquad \sup_{\substack{J \subset I \\ J \ni x}} \langle |f\mathbb{1}_I| \rangle_J \leq M_{\mathcal{D}}(f\mathbb{1}_I)(x)$$

$$\langle |f\mathbb{1}_I| \rangle_J = \frac{1}{|J|}\int_{I \cap J} |f| = \begin{cases} \langle |f| \rangle_J & \text{if } J \subset I \\ \frac{1}{|J|}\int_I |f| = \frac{|I|}{|J|}\langle |f| \rangle_I \leq \langle |f| \rangle_I & (\text{if } J \supsetneq I) \end{cases}$$

▨

→ Let: $\boxed{g(x) := \mathbb{1}_I(x)\left(f(x) - \langle f \rangle_{\hat{I}}\right)}$ where $\hat{I}$ is the dyadic parent of I.

$\Rightarrow M_{\mathcal{D}}g(x) \geq \underbrace{M_{\mathcal{D}}(f\mathbb{1}_I)(x)}_{>2\lambda} - \underbrace{M(\langle f \rangle_{\hat{I}}\mathbb{1}_I)(x)}_{|\langle f \rangle_{\hat{I}}| \leq \langle |f| \rangle_{\hat{I}} \leq \lambda}$ by sublinearity of $M_{\mathcal{D}}$; by maximality of I

$\Rightarrow M_{\mathcal{D}}g(x) > \lambda$

$M_{\mathcal{D}}(c\mathbb{1}_I)(x) = |c|, \forall x \in I$:

$x \in I \Rightarrow M_{\mathcal{D}}(c\mathbb{1}_I)(x) = \sup_{\substack{J \in \mathcal{D} \\ x \in J}} |c|\frac{|I \cap J|}{|J|}$

$= \sup_{\substack{J \cap I \neq \emptyset \\ J \ni x}} |c|\frac{|I \cap J|}{|J|}$ $\begin{cases} |c| & \text{if } J \subset I \\ |c|\frac{|I|}{|J|} \leq |c| & \text{if } J \supsetneq I \end{cases}$

$\Rightarrow |\{x \in I : M_{\mathcal{D}}f(x) > 2\lambda;\ f^{\#}(x) < \gamma\lambda\}| \leq |\{x \in I : M_{\mathcal{D}}f(x) > 2\lambda\}|$

$\leq |\{x \in I : M_{\mathcal{D}}g(x) > \lambda\}|$

$\leq \frac{1}{\lambda}\|g\|_1 = \frac{1}{\lambda}\int_I |f(y) - \langle f \rangle_{\hat{I}}|\,dy$

$\leq \frac{1}{\lambda}|\hat{I}| \underbrace{\frac{1}{|\hat{I}|}\int_{\hat{I}} |f(y) - \langle f \rangle_{\hat{I}}|\,dy}_{\leq f^{\#}(x),\ \forall x \in \hat{I}}$; choose $x = x_0$: $f^{\#}(x_0) < \gamma\lambda$

$\leq \frac{1}{\lambda}|\hat{I}| f^{\#}(x_0) < \gamma|\hat{I}|$

$= 2\gamma|I|$. //

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What about the claim $\boxed{\|f\|_p \lesssim \|f^{\#,p_0}\|_p, \ \forall 1<p<\infty}$?

The proof is essentially the same, only with a small modification at the end:

Show: $\boxed{|\{x: M_{\mathcal{D}}f(x)>2\lambda;\ f^{\#,p_0}(x)<\gamma\lambda\}| \le 2^{1/p_0}\gamma\,|\{x: M_{\mathcal{D}}f(x)>\lambda\}|}$, $\forall \lambda,\gamma>0$

Then by the RDI theorem:

$|\{x: M_{\mathcal{D}}f(x)>\lambda;\ f^{\#,p_0}(x)<\gamma\lambda/2\}| \le 2^{1/p_0}\gamma\,|\{x: M_{\mathcal{D}}f(x)>\frac{1}{2}\lambda\}|$

$\Rightarrow \int (M_{\mathcal{D}}f)^p \lesssim \int (f^{\#,p_0})^p$ as long as $2^{1/p_0}\gamma < \frac{1}{2^p}$, or $\gamma < 2^{-p-1/p_0}$ (choose as small as we wish)

$\Rightarrow \|f\|_p \le \|M_{\mathcal{D}}f\|_p \lesssim \|f^{\#,p_0}\|_p$.

→ $E_\lambda := \{M_{\mathcal{D}}f(x)>\lambda\}$ is the disjoint countable union of maximal dyadic $I\in\mathcal{J}_\lambda^{max}$ w/ $\langle|f|\rangle_I>\lambda$

$\Rightarrow$ Suffices to prove:

$\boxed{|\{x\in I: M_{\mathcal{D}}f(x)>2\lambda;\ f^{\#,p_0}(x)<\gamma\lambda\}| \le 2^{1/p_0}\gamma\,|I|, \ \forall I\in\mathcal{J}_\lambda^{max}}$

→ So let $I\in\mathcal{J}_\lambda^{max}$ and can assume there is $x_0\in I$ such that $f^{\#,p_0}(x_0)<\gamma\lambda$.

→ Still have: $x\in I$ s.t. $M_{\mathcal{D}}f(x)>2\lambda \Rightarrow M_{\mathcal{D}}(\mathbb{1}_I f)(x)>2\lambda$;

→ Let again $g(x)=\mathbb{1}_I(x)\,(f(x)-\langle f\rangle_{\hat{I}}) \Rightarrow M_{\mathcal{D}}g(x)>\lambda,\ \forall x\in I$ s.t. $M_{\mathcal{D}}f(x)>2\lambda$

$$
\begin{aligned}
\Rightarrow |\{x\in I: M_{\mathcal{D}}f(x)>2\lambda;\ f^{\#,p_0}(x)<\gamma\lambda\}| &\le |\{x\in I: M_{\mathcal{D}}g(x)>\lambda\}| \\
&\le \frac{1}{\lambda}\|g\|_1 = \frac{1}{\lambda}\int_I |f(y)-\langle f\rangle_{\hat{I}}|\,dy \\
&\le \frac{1}{\lambda}|\hat{I}|^{1/p_0}\underbrace{\left(\frac{1}{|\hat{I}|}\int_{\hat{I}}|f(y)-\langle f\rangle_{\hat{I}}|^{p_0}dy\right)^{1/p_0}}_{\le f^{\#,p_0}(x),\ \forall x\in\hat{I};\ \text{choose } x=x_0}|I|^{1/p_0'} \\
&\le \frac{1}{\lambda}2^{1/p_0}|I|^{1/p_0}|I|^{1/p_0'}f^{\#,p_0}(x_0) \\
&< \frac{1}{\cancel{\lambda}}2^{1/p_0}|I|\gamma\cancel{\lambda} = 2^{1/p_0}\gamma|I|.
\end{aligned}
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